MATHEMATICS FOR BUSINESS ECONOMICS

Boom

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THIRD EDITION

Mathematics for Business Economics

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Preface

This mathematics book is intended for starting (business) economics students. It is based on many years of experience in mathematics lectures to students of Tilburg University and on teaching high school students. The book repeats content of high school (elementary calculus, functions, derivatives), but also extends to new contents (partial derivatives, (constrained) optimization). The focus is on economic applications. There are many examples from microeconomics, modern portfolio theory, inventory management and statistics. In this new edition, a chapter is added that contains a large set of diagnostic exercises.

In addition to the book there is an **e-learning environment** where all mathematical topics are explained using text, multiple choice questions and films. The address of this e-learning environment is **www.wiskundeoptiu.nl/business-economics**.

The solutions of all exercises are available in the appendix.

This book would not have been written without the help of a large number of colleagues that supported us with tips and tricks. In particular, we want to mention Ruud Brekelmans, Bart Husslage. Elleke Janssen and Marieke Quant, who contributed to a great extend to the development of the e-learning environment. Moreover, additional thanks to Elleke for het support with LaTeX, in particular with respect to the lay-out and figures. Further, we thank Marieke Quant and Jop Schouten for their contribution to Chapter 7.

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Introduction

This introduction first presents a short overview of the contents of this book. Next, we discuss the structure of the book. Finally, some recommendations for studying are provided.

In Chapter 1 we introduce functions of one variable. Besides an overview of elementary functions, also the determination of zeros, solving inequalities and calculations using powers are discussed.

Chapter 2 is devoted to the derivative of a function of one variable. Besides a number of differentiation rules, also the inverse function and the economic notions of marginality and elasticity are presented.

In Chapter 3 we introduce functions of two variables and the notion of a level curve. Utility of a consumer and a first introduction to modern portfolio theory are examples of economic applications that will be addressed.

Partial derivatives and partial differentiation are the key elements of Chapter 4. Moreover, the economic notion of marginal rate of substitution is related to the tangent line to a level curve.

Chapter 5 is completely dedicated to the determination of extrema, either with or without constraints. As applications from economics we discuss, among other things, profit maximization of a producer, the Economic Order Quantity model, the regression line, utility maximization of a consumer, cost minimizing of a producer and optimal portfolio selection. Furthermore, also convex and concave functions are discussed, because these functions frequently appear in economic models.

Finally, in Chapter 6 the notion of an integral is introduced and used as tool to determine the area of a region. We apply integrals for the determination of consumer surplus and for the calculation of probabilities.

In this book mathematical and economic topics are discussed in separate sections. In the mathematical sections a mathematical notion or method is followed by a (mathematical) example and an exercise. The example is usually a complete solution or an illustration of the corresponding notion or method; the exercise is a way to fully capture a notion or a method. A similar remark can be made concerning the economic-oriented sections.

Each chapter is concluded with a section of diagnostic exercises. The answers to the exercises are included in the appendix.

The first step in learning mathematics is to visit the lectures that are offered at your university. Understanding mathematics can only be attained by trying to do the exercises yourselves. You will not always manage to solve an exercise. Then it is suggested that you study again the example directly before the exercise. If this provides insufficient information, you have to consult the theory in the book again. Another option is to use the e-learning environment that can be found at **www.wiskundeoptiu.nl/business-**

economics. Here the most important notions and methods are explained in an example by a teacher. One can also practice more with many multiple-choice questions. If you have finished all the exercises in the text, then you can start with the diagnostic exercises of that chapter. These exercises reveal whether you completely understand the chapter.

1
Functions of one variable

Functions are frequently used to describe relations between (economic) variables. In this chapter we introduce the notion of a function of one variable and provide an overview of the classes of polynomial functions, power functions, exponential functions and logarithmic functions. Moreover, elementary algebraic operations such as solving equations and inequalities are discussed.

List of notions

- Function of one variable
- Zero of a function
- Point of intersection of graphs of functions
- Linear function
- Quadratic function
- Polynomial function
- Power function
- Exponential function
- Logarithmic function

1.1 Introduction to functions of one variable

To describe the relation between variables we can use the mathematical notion of a function. In this section we discuss the notion of a function of one variable.

We start this section with an example in which we show how a function can be used to describe the relation between the cost and the quantity of a produced good.

Example 1.1: cost function

A manager has investigated a production process to analyze its costs. The result of this investigation is a relation between the production level q and the corresponding costs C, which can be displayed graphically in a coordinate system (see Figure 1.1). In this coordinate system the horizontal axis represents the production level and the vertical axis represents the corresponding costs. In other words, the independent variable (the production level) is displayed on the horizontal axis and the dependent variable (the costs) is displayed on the vertical axis.

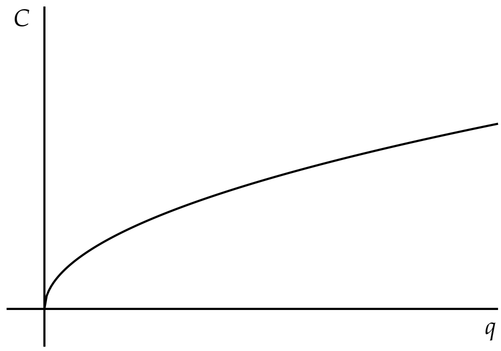

Figure 1.1 The cost function of a production process.

A graphical relation between the production level and the costs is usually not sufficient. A manager wants to know for each production level q the exact corresponding costs $C(q)$. Therefore, it is more useful to describe the relation between the production level and costs by a function: that is a prescription that calculates for each feasible production level q the corresponding costs C. The prescription that fits the graph of Figure 1.1 is $\sqrt{q}$. This prescription means that if the production level q is equal to 9, the costs are $\sqrt{9} = 3$, and if the production level q is equal to 16, the costs are $\sqrt{16} = 4$, etcetera.

Each time the costs are obtained by applying the *prescription* $\sqrt{q}$ to a value of the production level q. Since we assign no interpretation to production levels with a negative value, we restrict the prescription to values q greater than or equal to 0. We conclude that the variables C and q satisfy the equation $C = \sqrt{q}$. We may say that 'the costs C is a function of the production level q'. ◀

If the costs C is a function of the production level q and we do not specify the actual prescription, then we denote the prescription by $C(q)$. The relation between C and q is then given by the equation $C = C(q)$. Hence, in the above example $C(q)$ represents $\sqrt{q}$.

Functions of one variable

A *function of the variable* x is a prescription $y(x)$ which calculates a number, the function value, for any feasible value of the variable x.
The set of all feasible values D of x is called the *domain* of the function.
The set of all possible function values is called the *range* of the function.

If the domain D of a function $y(x)$ is not explicitly given, then the domain consists of all x for which the prescription $y(x)$ can be executed. If additional restrictions are provided, then this will be explicitly stated. Such a restriction, like non-negativity of x, can for example arise from the economic interpretation of the model.

The function values $y(x)$ can be interpreted as the values of a variable. If we call this variable y, then y and x satisfy the equation

$$y = y(x).$$

The variable x in $y(x)$ is called the *independent* or *input variable* and the variable y is called the *dependent* or *output variable*.

Functions and variables are denoted by letters or combinations of letters. Maybe you are used to denoting variables by the letters x, y or z and a function by the letter f (hence $f(x)$ instead of $y(x)$, so that $y = f(x)$). In economic theory, however, one often chooses a letter or a combination of letters that is close to the meaning of the variable (p for **p**rice, L for **L**abour, K (from German '**K**apital') for capital, w for **w**age, etcetera), or of the prescription (MC for **M**arginal **C**ost function, AC for **A**verage **C**ost, etcetera).

The *graph* of a function $y(x)$ is a graphical representation of the function in a coordinate system with two axes, the x-axis and the y-axis, consisting of points with coordinates $(x, y(x))$. It is common to use the horizontal axis for the independent variable x and the vertical axis for the dependent variable y.

Example 1.2: graph of a function

In Figure 1.2 the graph of the function $y(x) = 1 + \sqrt{x+2}$ is shown. The domain of $y(x)$ is $x \geq -2$ and the range is $y \geq 1$. In interval notation the domain is $[-2, \infty)$ and the range $[1, \infty)$.

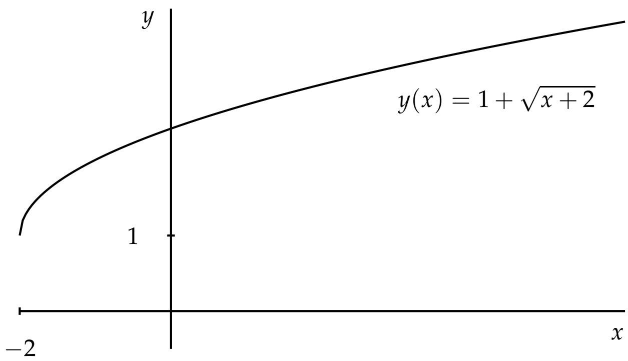

Figure 1.2 *The graph of the function* $y(x) = 1 + \sqrt{x+2}$ *in an* (x, y)*-coordinate system.*

A point of intersection of the graph of a function $y(x)$ with the x-axis can be determined by calculating a zero of the function $y(x)$.

Zero of a function of one variable

A *zero* of a function $y(x)$ is a solution of the equation $y(x) = 0$.

A zero a of the function $y(x)$ gives the point of intersection $(a, 0)$ of the corresponding graph and the x-axis.

Point of intersection of two graphs

A point of intersection of the graph $y(x)$ with the graph of another function $z(x)$ is a point (a, b) where a is a solution of the equation $y(x) = z(x)$ and $b = y(a) (= z(a))$.

For the determination of a point of intersection of the graphs of the functions $y(x)$ and $z(x)$ we first calculate the x-coordinate by solving the equation

$$y(x) = z(x).$$

Subsequently, the y-coordinate is obtained by substituting that x-coordinate into one of the two functions.

A point of intersection of the graph of a function $y(x)$ with the y-axis is obtained by calculating the function value at $x = 0$. Hence, a point of intersection with the y-axis is $(0, y(0))$.

Example 1.3: zero and point of intersection

Consider the functions $y(x) = -2x + 2$ and $z(x) = x - 4$. The zero of $y(x)$ is the solution of $y(x) = 0$,

$$\begin{aligned} y(x) = 0 \quad &\Leftrightarrow \quad -2x + 2 = 0 \\ &\Leftrightarrow \quad -2x = -2 \\ &\Leftrightarrow \quad x = 1. \end{aligned}$$

Hence, the point $(1, 0)$ is the point of intersection of the graph of $y(x)$ with the x-axis. A point of intersection of the graph of $y(x)$ with the graph of $z(x)$ is obtained by solving the equation $y(x) = z(x)$,

$$\begin{aligned} y(x) = z(x) \quad &\Leftrightarrow \quad -2x + 2 = x - 4 \\ &\Leftrightarrow \quad -3x = -6 \\ &\Leftrightarrow \quad x = 2. \end{aligned}$$

The x-coordinate of the point of intersection is $x = 2$. Substituting $x = 2$ into the function $y(x)$ we get the y-coordinate, which equals $y(2) = -2$. Obviously, we also have $z(2) = -2$. The point of intersection is $(2, -2)$. Since $y(0) = 2$, the graph of $y(x)$ intersects the y-axis at the point $(0, 2)$.

In the upcoming sections of this chapter we discuss several elementary functions of one variable, show the corresponding graphs, provide some properties and calculate zeros and points of intersection.

1.2 Overview of functions of one variable

1.2.1 Polynomial functions

In this subsection we start with an overview of constant, linear and quadratic functions. Subsequently, we provide the general expression of a polynomial function. In this overview the solving of equations and inequalities is also discussed.

Constant functions

A function of the form

$$y(x) = c,$$

where c is a number, is called a *constant function*. A constant function has the same value for each x. Henceforth, the graph of a constant function is a horizontal line.

Example 1.4: graph of a constant function

An example of a constant function is the function $y(x) = 3$ and its graph is shown in Figure 1.3.

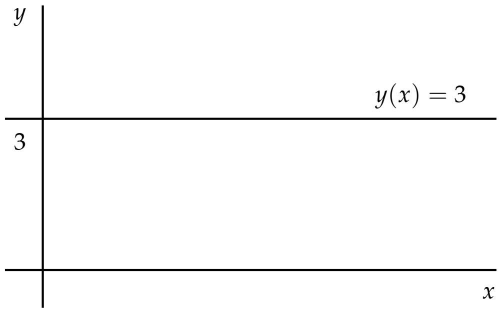

Figure 1.3 The graph of the constant function $y(x) = 3$.

Note that a constant function $y(x) = c$ has no zeros if $c \neq 0$ and that every x is a zero if $c = 0$.

Linear functions

A function of the form

$$y(x) = ax + b,$$

where a and b are numbers ($a \neq 0$), is called a *linear function*. Note that if $a = 0$ then the function $y(x)$ is a constant function. The graph of a linear function is a straight line. The *slope* of the line is equal to the number a. The slope of the graph of a linear function indicates the change of the function value if the input x increases by one unit. Since for a linear function this change is always equal to a, it holds that for any x

$$y(x + 1) - y(x) = a.$$

A linear function has a positive slope if $a > 0$ and a negative slope if $a < 0$. Note that the slope of a constant function is equal to 0.

Example 1.5: graph of a linear function

The graphs of the linear functions $y(x) = 3x + 2$ and $z(x) = -2x + 1$ are shown in Figure 1.4 in an (x, y)-coordinate system. The slope of the graph of $y(x)$ is equal to 3 and the slope of the graph of $z(x)$ is equal to -2.

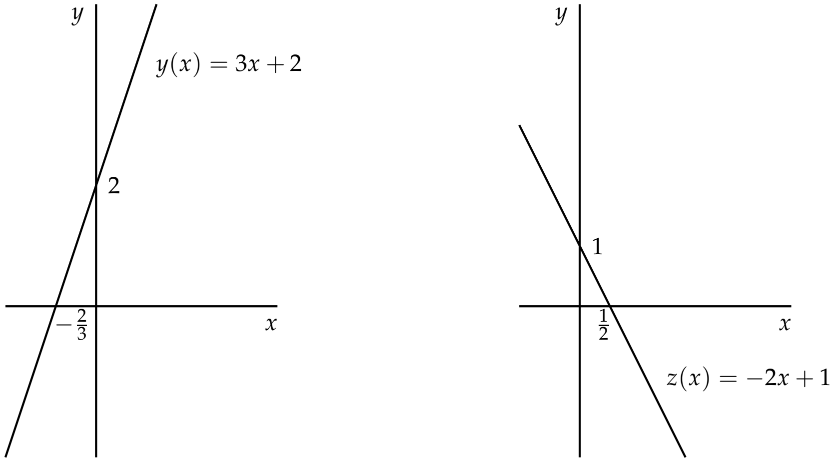

Figure 1.4 *The graph of the function $y(x) = 3x + 2$ (left).*
The graph of the function $z(x) = -2x + 1$ (right).

In the final example of Section 1.1 we have calculated zeros and points of intersection for linear functions. It is straightforward to determine a general expression for the point of intersection of the graph of a linear function and the x-axis or y-axis, respectively. To determine the point of intersection of the graph of a linear function $y(x) = ax + b$ with the x-axis, we calculate the zero of the function $y(x)$,

$$\begin{aligned} y(x) = 0 \quad &\Leftrightarrow \quad ax + b = 0 \\ &\Leftrightarrow \quad ax = -b \\ &\Leftrightarrow \quad x = -\frac{b}{a}. \end{aligned}$$

Since a linear function $y(x) = ax + b$ has precisely one zero, the graph has precisely one point of intersection with the x-axis. Hence, the point of intersection of the graph of a linear function and the x-axis is $(-\frac{b}{a}, 0)$. The point of intersection with the y-axis is obtained by substituting $x = 0$ into the function: $y(0) = b$. Hence, the point of intersection with the y-axis is $(0, b)$. We can conclude that for a linear function $y(x) = ax + b$ the slope of the line is represented by a, the zero equals $-\frac{b}{a}$ and b is the y-coordinate of the point of intersection of the graph with the y-axis.

Exercise 1.1 (zero and point of intersection)

Consider the functions $y(x) = 3x + 2$ and $z(x) = -5x + 4$.

a) Draw the graphs of the functions $y(x)$ and $z(x)$.
b) Determine the zero of each of the two functions.
c) Determine for the graph of each function the point of intersection with the x-axis.
d) Determine for the graph of each function the point of intersection with the y-axis.
e) Determine the point of intersection of the graphs of these two functions.

Exercise 1.2 (slope of a line)

The points $(2,4)$ and $(3,9)$ are on the graph of a linear function $y(x) = ax + b$. Determine a and b.

Quadratic functions

A function of the form

$$y(x) = ax^2 + bx + c,$$

where a, b and c are numbers ($a \neq 0$) is called a *quadratic function*. Note that if $a = 0$ the function is either linear (if $b \neq 0$) or constant (if $b = 0$). The graph of the function $y(x) = ax^2 + bx + c$ is a parabola. This parabola opens upward if $a > 0$ and opens downward if $a < 0$.

Example 1.6: graphs of quadratic functions

The graphs of the quadratic functions $y(x) = -x^2 + 2x + 3$ and $z(x) = 2x^2 + 1$ are shown in Figure 1.5. The graph of the function $y(x)$ is a parabola opened downward because the coefficient of x^2 is equal to -1 (negative), and the graph of the function $z(x)$ is a parabola opened upward because the coefficient of x^2 is equal to 2 (positive).

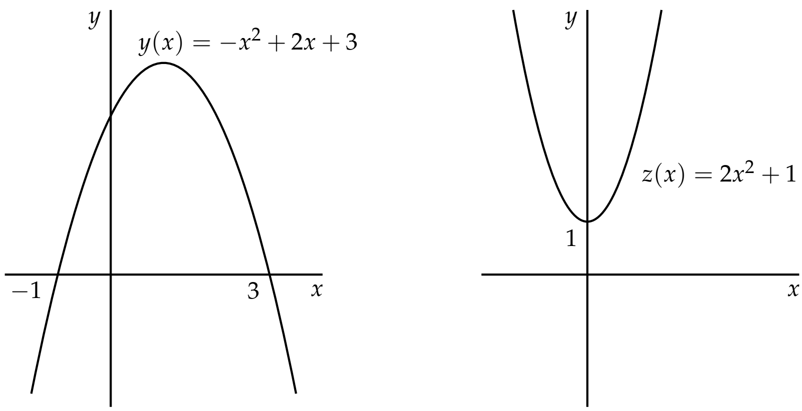

Figure 1.5 The graph of $y(x) = -x^2 + 2x + 3$ is a parabola opened downward (left). The graph of $z(x) = 2x^2 + 1$ is a parabola opened upward (right).

The points of intersection of the graph of a quadratic function $y(x) = ax^2 + bx + c$ and the x-axis are determined by solving the quadratic equation

$$ax^2 + bx + c = 0.$$

A quadratic equation can be solved using the *quadratic formula*. In this formula the *discriminant* is a key component.

The discriminant of a quadratic equation $ax^2 + bx + c = 0$ is equal to $b^2 - 4ac$ and is denoted by D,

$$D = b^2 - 4ac.$$

A quadratic equation has either two, one or zero solutions, depending on the value of the discriminant. The solutions of a quadratic equation are displayed in the following discriminant criterion.

Discriminant criterion and quadratic formula of a quadratic equation

For a quadratic equation $ax^2 + bx + c = 0$ with $a \neq 0$, the following holds for $D = b^2 - 4ac$:

(i) if $D > 0$, then the solutions of the quadratic equation are

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

(ii) if $D = 0$, then the solution of the quadratic equation is

$$x = -\frac{b}{2a}.$$

(iii) if $D < 0$, then the quadratic equation has no solutions.

Example 1.7: zeros of a quadratic function

Consider the function $y(x) = -x^2 + 2x + 3$. A zero of $y(x)$ is a solution of the equation

$$-x^2 + 2x + 3 = 0.$$

The discriminant is $D = 2^2 - 4 \cdot -1 \cdot 3 = 16$. Because $D > 0$ we obtain according to the quadratic formula the following two solutions:

$$x = \frac{-2 + \sqrt{2^2 - 4 \cdot -1 \cdot 3}}{2 \cdot -1} = -1 \quad \text{and} \quad x = \frac{-2 - \sqrt{2^2 - 4 \cdot -1 \cdot 3}}{2 \cdot -1} = 3.$$

The points of intersection of the graph with the x-axis are $(-1, 0)$ and $(3, 0)$, as can be seen in Figure 1.5.

Exercise 1.3 (zeros of a quadratic function)

Determine the zeros of the following quadratic functions:

a) $y(x) = x^2 + 7x + 6$. b) $y(x) = 4x^2 + 2x + 1$.

Exercise 1.4 (graph of a quadratic function)

Consider the two functions $y(x) = x^2 + 4x + 3$ and $z(x) = -x^2 + 6$.

a) Sketch the graphs of the functions $y(x)$ and $z(x)$.

b) Determine the points of intersection of the graphs of these functions.

When considering two functions the points of intersection are not all that is interesting: also the values of x where the value of one function is greater than or less than the value of the other function could be relevant. To solve this kind of problem we have to solve inequalities. To solve the inequality $f(x) \geq g(x)$ we use the following step plan.

Solving inequality $f(x) \geq g(x)$

Step 1.	Define the function $h(x) = f(x) - g(x)$.
Step 2.	Determine the zeros of $h(x)$.
Step 3.	Make a sign chart of $h(x)$.
Step 4.	Observe in the sign chart where $h(x) \geq 0$.
Conclusion:	The values where $h(x) \geq 0$ are identical to the values where $f(x) \geq g(x)$.

Obviously, we can solve the inequalities $f(x) > g(x), f(x) \leq g(x)$ and $f(x) < g(x)$ in a similar way. In the following example we illustrate the step plan for solving an inequality.

Example 1.8: solving inequalities

Consider the functions $f(x) = x^2 + 2$ and $g(x) = -3x$. We want to find the values of x where $f(x) \geq g(x)$.

Step 1. Define the function $h(x) = f(x) - g(x)$.
We get $h(x) = f(x) - g(x) = x^2 + 2 - (-3x) = x^2 + 3x + 2$.

Step 2. Determine the zeros of $h(x)$.

$$\begin{aligned} h(x) = 0 \quad &\Leftrightarrow \quad x^2 + 3x + 2 = 0 \\ &\Leftrightarrow \quad x = -1 \text{ or } x = -2, \end{aligned}$$

where the solutions are obtained by using the quadratic formula.

Step 3. Make a sign chart of $h(x)$.
The sign chart of $h(x)$ is obtained by putting the zeros on a straight line. This divides the line into three intervals: $(-\infty, -2), (-2, -1)$ and $(-1, \infty)$. The function values of $h(x)$ in one interval all have the same sign. This implies that either the function values are all positive or all negative in one interval. Hence, by choosing an arbitrary value in $(-\infty, -2)$ we can determine the sign of $h(x)$ in this interval. For instance, choose $x = -3$. Then $h(-3) = (-3)^2 + 3 \cdot (-3) + 2 = 2 > 0$. Hence, we can conclude that $h(x) > 0$ for all x in the interval $(-\infty, -2)$. Since $h(-1.5) = -0.25$ it holds that $h(x) < 0$ for all x in the interval $(-2, -1)$. Similarly, we find that $h(x) > 0$ for all x in the interval $(-1, \infty)$, because $h(0) = 2$. Figure 1.6 shows the sign chart of $h(x)$.

Mathematics for Business Economics is intended for starting (business) economics students. The book repeats high school mathematics, but also extends to new domains. The focus is on economic applications: throughout the book many examples are given from microeconomics, modern portfolio theory, inventory management and statistics. A recurrent theme in these examples is the mathematical technique of optimization.

For a better understanding this book contains a large number of exercises. Furthermore, there is a freely accessible e-learning environment, where all mathematical topics are explained using text, multiple choice questions and films. Together, the book and e-learning environment provide a solid mathematical foundation for your study (business) economics.

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